

Now let's apply the Boltzmann equation (~~or rather its~~  
~~Saha equation approximation~~)

## Big Bang Nucleosynthesis

$T \sim 1 \text{ MeV}$  (typical nuclear binding energies)

Plasma consists of:

- relativistic particles in equilibrium

$$\gamma, e, e^+ \quad e^+e^- \leftrightarrow \gamma\gamma$$

- decoupled relativistic neutrinos

$$\nu e \leftrightarrow \bar{\nu} e \quad \text{decouples at } T \sim 1 \text{ MeV}$$

- non-relativistic baryons

$p, n$

They have not annihilated only thanks to a primordial asymmetry

$$\frac{n_b - n_{\bar{b}}}{s} \sim 10^{-10}$$

At  $T \sim 1 \text{ MeV}$   $n_b \equiv \frac{n_b}{n_\gamma} \sim 10^{-10}$  baryon-to-photon ratio

Many more  $\gamma$  than baryons!

BBN tells us how  $p, n$  end up: not Fe, but lighter nuclei, thanks to nuclear reactions

Simplifications:

• track H and He only (and their isotopes):  $^1\text{H}, ^2\text{H}, ^3\text{H}, ^3\text{He}, ^4\text{He}$

•  $T \gtrsim 0.1 \text{ MeV}$ : only free  $n, p$

↳ solve for  $\frac{n}{p}$  ratios, use this as input for synthesis of  $^4\text{He}$

Are these good approximations? Take e.g. Deuterium production



Saha equation (considering  $\gamma$  in equilibrium  $\rightarrow n_\gamma = n_\gamma^{(0)}$ )

$$\frac{n_D}{n_n n_p} = \frac{n_D^{(0)}}{n_n^{(0)} n_p^{(0)}}$$

$$n_i = g_i \left( \frac{m_i T}{2\pi} \right)^{3/2} e^{-\frac{m_i}{T}}$$

since  $m_i \gg T$

$$g_i = 3(D), 2(n), 2(p)$$

$$\Rightarrow \frac{n_D}{n_n n_p} = \frac{3}{4} \left( \frac{2\pi m_D}{m_n m_p T} \right)^{3/2} e^{-\frac{m_n + m_p - m_D}{T}}$$

$B_D \equiv m_n + m_p - m_D \approx 2.22$  MeV binding energy of Deuterium

$$\Rightarrow \frac{n_D}{n_n n_p} \approx \frac{3}{4} \left( \frac{4\pi}{m_p T} \right)^{3/2} e^{-\frac{B_D}{T}}$$

$m_D \approx 2m_n \approx 2m_p$  in prefactor

$$\frac{n_n, n_p}{n_b} \sim n_b = n_\gamma \eta_b \sim \eta_b T^3 \Rightarrow \frac{n_D}{n_n n_p} \sim \frac{n_D}{n_b \eta_b T^3}$$

$$\Rightarrow \frac{n_D}{n_b} \sim \eta_b \left( \frac{T}{m_p} \right)^{3/2} e^{-\frac{B_D}{T}}$$

Small!

Small, dominated by pre-factor unless  $e^{-\frac{B_D}{T}}$  large!

As until  $T \ll B_D$ , both LHS and RHS tiny  $\rightarrow \frac{n_D}{n_b} \ll 1$

$\rightarrow$  deuterium cannot be produced, only free p and n

A  $T \sim$  MeV BBN starts, produces elements up to  ${}^4\text{He}$

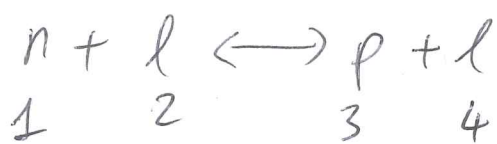
Strategy: determine  $\frac{n_n}{n_p}$ , use it as input for subsequent reactions

Neutron fraction  $X_n \equiv \frac{n_n}{n_n + n_p} \xrightarrow{\text{equilibrium}} X_{n,eq} \equiv \frac{1}{1 + \frac{n_p^{(0)}}{n_n^{(0)}}}$

Essentially we want to track  $X_n$  down to  $\sim 1$  MeV

Boltzmann equation:

$$\frac{1}{a^3} \frac{d(n_i a^3)}{dt} = n_1^{(0)} n_2^{(0)} \langle \sigma v \rangle \left( \frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}} \right)$$



$$l = \nu_e, e, \bar{\nu}_e, e^+$$

in equilibrium!  $n_e = n_{e^+}^{(0)}$

$$\frac{1}{a^3} \frac{d(n_n a^3)}{dt} = n_e^{(0)} n_n^{(0)} \langle \sigma v \rangle \left( \frac{n_p n_e}{n_p^{(0)} n_e^{(0)}} - \frac{n_n n_e}{n_n^{(0)} n_e^{(0)}} \right) =$$

$$= n_e^{(0)} \langle \sigma v \rangle \left( \frac{n_p n_n^{(0)}}{n_p^{(0)}} - n_n \right)$$

$$\frac{n_n^{(0)}}{n_p^{(0)}} = \left( \frac{m_n}{m_p} \right) e^{-\frac{(m_n - m_p)}{T}} \equiv \left( \frac{m_n}{m_p} \right) e^{-\frac{Q}{T}} \approx e^{-\frac{Q}{T}}$$

$\gg 1$

$$Q \equiv m_n - m_p \approx 1.3 \text{ MeV}$$

$n_e^{(0)} \langle \sigma v \rangle \equiv \lambda_{np}$  rate for neutron-proton conversion

$$n_n = (n_n + n_p) X_n \rightarrow \cancel{a^{-3} \frac{d(n_n a^3)}{dt}} = \cancel{a^{-3} \frac{d[(n_n + n_p) X_n a^3]}{dt}}$$

$$\rightarrow a^{-3} \frac{d(n_n a^3)}{dt} = a^{-3} \frac{d[(n_n + n_p) a^3 X_n]}{dt} = \underbrace{a^{-3} (n_n + n_p) a^3}_{(n_n + n_p) a^3 \sim n_b a^3 = \text{const}} \frac{dX_n}{dt} = (n_n + n_p) \frac{dX_n}{dt}$$



$$(n_n + n_p) \frac{dX_n}{dt} = \lambda_{np} \left( \frac{n_p n_n^{(0)}}{n_p^{(0)}} - n_n \right) = \lambda_{np} \left( n_p e^{-\frac{Q}{T}} - n_n \right)$$

$$\Rightarrow \left[ \frac{dX_n}{dt} = \lambda_{np} \left( \frac{n_p}{n_n + n_p} e^{-\frac{Q}{T}} - \frac{n_n}{n_n + n_p} \right) = \lambda_{np} \left[ (1 - X_n) e^{-\frac{Q}{T}} - X_n \right] \right]$$

$$\boxed{\frac{dX_n}{dt} = \lambda_{np} \left[ (1 - X_n) e^{-\frac{Q}{T}} - X_n \right]}$$

note:  $T, \lambda_{np} = T(t), \lambda_{np}(t)!$

New variable  $x \equiv \frac{Q}{T}$

$$\frac{dX_n}{dt} = \frac{dX_n}{dx} \frac{dx}{dt} = -x \frac{\dot{T}}{T} \frac{dX_n}{dx}$$

$$\frac{dX_n}{dt} = \frac{dX_n}{dx} \frac{dx}{dt} = \frac{dX_n}{dx} \times \frac{\dot{T}}{T} = x H \frac{dX_n}{dx}$$

$$\frac{dx}{dt} = \frac{d}{dt} \left( \frac{Q}{T} \right) = -\frac{Q}{T^2} \dot{T} = -\frac{Q}{T} \frac{\dot{T}}{T} = -x \frac{\dot{T}}{T}$$

$$\frac{dX_n}{dt} = x H \frac{dX_n}{dx} \quad H = \sqrt{\frac{8\pi G \rho}{3}} \quad \rho = \frac{\pi^2}{30} T^4 \left[ \sum_{\text{bosons}} g_i + \frac{7}{8} \sum_{\text{fermions}} g_j \right]$$

$$= g_*(T) \frac{\pi^2}{30} T^4$$

at  $T \sim \text{MeV}$  relativistic particles:  $\gamma$  ( $g_\gamma = 2$ ),  $\nu$  ( $g_\nu = 6$ ),  $e^\pm$  ( $g_{e^+} = g_{e^-} = 2$ )

$\Rightarrow g_* \approx 10.75$  constant in  $T \sim \text{MeV}$

$$x H \frac{dX_n}{dx} = x \frac{dX_n}{dx} \sqrt{\frac{8\pi G}{3} g_* \frac{\pi^2}{30} T^4} = x T^2 \frac{dX_n}{dx} \sqrt{\frac{8\pi G g_* \pi^2}{3 \cdot 30}} = x \frac{Q^2}{x^2} \sqrt{\frac{4\pi^3 G}{45}} \times \sqrt{10.75} =$$

$$= \frac{1}{x} \left( \frac{4\pi^3 G Q^4}{45} \times \sqrt{10.75} \right) \Rightarrow H(T=Q) = H(x=1) \approx 1.1 \text{ s}^{-1}$$

$$\frac{dX_n}{dt} = \lambda_{np} \left[ (1-X_n) e^{-\frac{Q}{T}} - X_n \right]$$

$$\hookrightarrow \frac{H(x=1)}{x} \frac{dX_n}{dx} = \lambda_{np} \left[ (1-X_n) e^{-\frac{Q}{T}} - X_n \right]$$

$$\hookrightarrow \frac{dX_n}{dx} = \frac{x \lambda_{np}}{H(x=1)} \left[ (1-X_n) e^{-x} - X_n \right]$$

$$= e^{-x} - X_n e^{-x} - X_n = e^{-x} - X_n (1+e^{-x})$$

$$\boxed{\frac{dX_n}{dx} = \frac{x \lambda_{np}}{H(x=1)} \left[ e^{-x} - X_n (1+e^{-x}) \right]}$$

$$\lambda_{np} \approx \frac{255}{\tau_n x^5} (12 + 6x + x^2)$$

$$\tau_n \approx 885.7 \text{ s}$$

$$T \sim \text{MeV} \quad x \sim 1 \rightarrow \lambda_{np} \sim 5.5 \text{ s}^{-1}, \quad H \sim 1.1 \text{ s}^{-1}$$

$\lambda_{np} > H \Rightarrow$  still equilibrium

Integrate  $\frac{dX_n}{dx}$  numerically

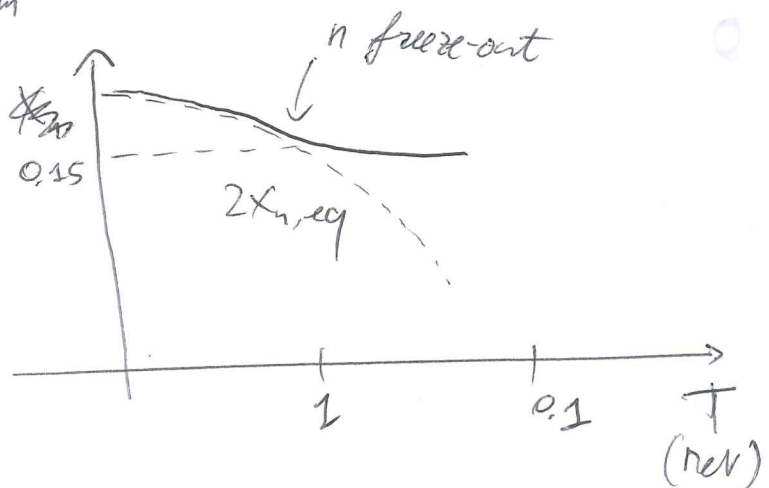
For  $T \lesssim 0.5 \text{ MeV}$ ,  $X_n$  freezes out at  $X_n \sim 0.15$

Below 0.1 MeV need to include:

- neutron decay  $n \rightarrow p + e^- + \bar{\nu}_e$
- deuterium production  $n + p \rightarrow D + \gamma$

Production of D, He only really starts at  $T_{\text{me}} \sim 0.07 \text{ MeV}$

Neutron fraction depleted by  $e^{-\frac{t}{\tau_n}} \rightarrow X_n(T_{\text{me}}) \approx 0.11$



So nucleosynthesis starts with an input neutron fraction

$$X_n \approx 0.11$$

$$\frac{n_D}{n_b} \sim \eta_b \left(\frac{T}{m_p}\right)^{3/2} e^{-\frac{B_D}{T}} \rightarrow \ln(\eta_b) + \frac{3}{2} \ln\left(\frac{T_{nuc}}{m_p}\right) \sim -\frac{B_D}{T_{nuc}}$$

$T_{nuc} \sim 0.07 \text{ MeV}$

~~B~~ B higher for  ${}^4\text{He}$  than D  $\rightarrow$   ${}^4\text{He}$  production favored  
 So at  $T \approx T_{nuc}$  all remaining neutrons processed into  ${}^4\text{He}$

Need 2 neutrons for  ${}^4\text{He} \rightarrow n_{\text{He}} = \frac{n_n}{2}$

Mass fraction

~~$$Y_p \equiv \frac{4 n_{\text{He}}}{n_b} = 2 X_n(T_{nuc}) \approx 0.22$$~~

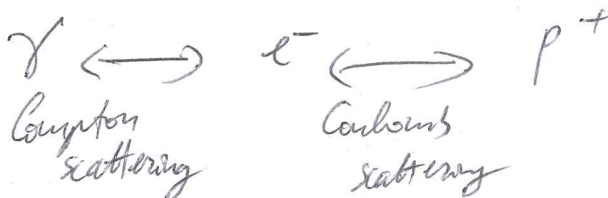
$$Y_p \approx 0.2262 + 0.0135 \ln\left(\frac{\eta_b}{10^{-10}}\right) \quad \text{Logarithmic sensitivity to } \Omega_b$$

$\rightarrow$  Excellent agreement with observations!  
 Pillar of observational cosmology

Besides  ${}^4\text{He}$ , BBN produces some D (excellent indicator of  $\Omega_b^{(2)}$ )  
 and  ${}^7\text{Li}$  (disagreement theory - observations)

Recombination

$$T \approx 1 \text{ eV}$$



All tightly coupled  $\rightarrow$  no neutral Hydrogen

$B \approx 13.6 \text{ eV}$ , but high  $\eta$  delays formation of H  
 until  $T \approx 0.25 \text{ eV}$