Without recombination $k_D \propto \sqrt{\Omega_b}$

Damping of anisotropies due to photon diffusion not simply due to finite thickness of last-scattering surface, because even before recombination $\dot\tau < \infty$. ~~Even if recombination instantaneous~~

Inhomogeneities to anisotropies

We want to express $\Theta_\ell(k,\eta_0)$ as a function of $\Theta_0(k,\eta_*)$ and $\Theta_1(k,\eta_*)$, then relate these to the CMB C_ℓs.

First we need to solve for $\Theta_\ell(\eta_0)$ in terms of $\Theta_{0,1}(k,\eta_*)$

Consider photon Boltzmann equation

$$\dot\Theta + ik\mu\Theta = -\dot\Phi - ik\mu\Psi - \dot\tau\left[\Theta_0 - \Theta + \mu v_b - \frac{1}{2}P_2(\mu)\Pi\right]$$

$$-\dot\tau\Theta \Rightarrow \boxed{\dot\Theta + ik\mu\Theta - \dot\tau\Theta}^{\,*} = -\dot\Phi - ik\mu\Psi - \dot\tau\left[\Theta_0 + \mu v_b - \frac{1}{2}P_2(\mu)\Pi\right]$$

$$* \quad \dot\Theta + ik\mu\Theta - \dot\tau\Theta = \dot\Theta + (ik\mu - \dot\tau)\Theta = e^{-ik\mu\eta+\tau}\frac{d}{d\eta}\left[\Theta e^{ik\mu\eta-\tau}\right]$$

check $$e^{-ik\mu\eta+\tau}\frac{d}{d\eta}\left[\Theta e^{ik\mu\eta-\tau}\right] = e^{-ik\mu\eta+\tau}\left[\dot\Theta e^{ik\mu\eta-\tau} + (ik\mu-\dot\tau)\Theta e^{ik\mu\eta-\tau}\right] =$$
$$= \dot\Theta + (ik\mu - \dot\tau)\Theta$$

So we can rewrite photon equation as

$$e^{-ik\mu\eta+\tau}\frac{d}{d\eta}\left[\Theta e^{ik\mu\eta-\tau}\right] = \tilde S$$

where source function defined as

$$\tilde S \equiv -\dot\Phi - ik\mu\Psi - \dot\tau\left[\Theta_0 + \mu v_b - \frac{1}{2}P_2(\mu)\Pi\right]$$

$$\Rightarrow \frac{d}{d\eta}\left[\Theta e^{ik\mu\eta - \tau}\right] = \tilde{S} e^{ik\mu\eta - \tau(\eta)}$$

$$\cancel{\Rightarrow \Theta e^{ik\mu\eta - \tau} = \int_{\eta_i}^{\eta_0} d\eta\, \tilde{S}(\eta) e^{ik}}$$

~~Integrate to get $\Theta(\eta_0)$ [from η_i to η_0]~~

$$\cancel{\Theta(\eta_0) = \Theta(\eta_i) e^{ik\mu(\eta_i - \eta_0)} e^{-\tau(\eta_i - \eta_0)} + \int_{\eta_i}^{\eta} d\eta\, \tilde{S}(\eta) e^{ik\mu(\eta-\eta_0) - \tau(\eta - \eta_0)}}$$

~~Integrate to get $\Theta(\eta_0)$ from η_i to η~~

$$\cancel{\Theta e^{ik\mu\eta - \tau(\eta)} = \int_{\eta_i}^{\eta} d\eta\, \tilde{S}(\eta) e^{ik\mu}}$$

Integrate to get $\Theta(\eta_0)$

$$\Theta(\eta_0) = \Theta(\eta_i) e^{ik\mu(\eta_i - \eta_0)} e^{-\tau(\eta_i)} + \int_{\eta_i}^{\eta_0} d\eta\, \tilde{S}(\eta) e^{ik\mu(\eta - \eta_0) - \tau(\eta)}$$

[using $\tau(\eta_0) = 0$ since $\tau \equiv \int_{\eta}^{\eta_0} d\eta'\, n_e \sigma_T a(\eta)$]

If $\eta_i \ll \eta_0$, τ very large at η_i so anything multiplied by $e^{-\tau(\eta_i)}$ can be neglected. Physically: Compton scattering erases initial anisotropies, so we may actually set $\eta_i \sim 0$

$$\Rightarrow \boxed{\Theta(k, \mu, \eta_0) \simeq \int_0^{\eta_0} d\eta\, \tilde{S}(k, \mu, \eta) e^{ik\mu(\eta - \eta_0) - \tau(\eta)}}$$

$\tilde{S}$ (which also depends on μ) implicitly hides all the complication

If $\tilde{S}$ did not depend on μ this could immediately become an equation for each of the Θ_ℓ s. Let's pretend this is the case... (forget for the moment μ-dependence of $\tilde{S}$)

$$\underbrace{\int_{-1}^{1}\frac{d\mu}{2}P_\ell(\mu)\,\Theta(k,\mu,\eta_0)}_{=(-i)^\ell\Theta_\ell} \simeq \int_{-1}^{1}\frac{d\mu}{2}P_\ell(\mu)\int_0^{\eta_0}d\eta\,\tilde{S}(k,\eta)e^{ik\mu(\eta-\eta_0)-\tau(\eta)}$$

$$= \int_0^{\eta_0}\underbrace{\left[\int_{-1}^{1}\frac{d\mu}{2}P_\ell(\mu)e^{ik\mu(\eta-\eta_0)}\right]}_{\frac{1}{(-i)^\ell}j_\ell[k(\eta-\eta_0)]}\tilde{S}(k,\eta)e^{-\tau(\eta)}$$

j_ℓ spherical Bessel function

$$\Rightarrow (-i)^\ell\Theta_\ell \simeq \frac{1}{(-i)^\ell}\int_0^{\eta_0}d\eta\,\tilde{S}(k,\eta)e^{-\tau(\eta)}j_\ell[k(\eta-\eta_0)]$$

$$\Rightarrow \boxed{\Theta_\ell(k,\eta_0) \simeq (-1)^\ell\int_0^{\eta_0}d\eta\,\tilde{S}(k,\eta)e^{-\tau(\eta)}j_\ell[k(\eta-\eta_0)]}$$

But $\tilde{S}$ actually depends on μ. Note however that we can "replace" $\mu \longrightarrow \frac{1}{ik}\frac{d}{d\eta}$ in $\tilde{S}$ since it multiplies $e^{ik\mu(\eta-\eta_0)}$

$$\tilde{S} = \dot{\Phi} - ik\mu\Psi - \dot{\tau}\left[\Theta_0 + \mu v_e - \frac{1}{2}P_2(\mu)\Pi\right]$$ e.g. consider $-ik\mu$

Explicitly

$$-ik\int_0^{\eta_0}d\eta\,\mu\Psi e^{ik\mu(\eta-\eta_0)}e^{-\tau(\eta)} = -\int_0^{\eta_0}d\eta\,\Psi e^{-\tau(\eta)}\underbrace{\frac{d}{d\eta}\left[e^{ik\mu(\eta-\eta_0)}\right]}_{=ik\mu e^{ik\mu(\eta-\eta_0)}} =$$

$$= -\Psi e^{-\tau(\eta)}e^{ik\mu(\eta-\eta_0)}\Big|_0^{\eta_0} + \int_0^{\eta_0}d\eta\,e^{ik\mu(\eta-\eta_0)}\frac{d}{d\eta}\left[\Psi e^{-\tau(\eta)}\right] =$$

$$\cancel{= +\Psi(0) e^{-\tau(0)} e^{ik\mu\eta}} \quad \tau(0) \text{ very large}$$

$$= \Psi(\eta_0) e^{-\tau(\eta_0)} - \cancel{\Psi(0) e^{-\tau(0)} e^{-ik\mu\eta_0}} + \int_0^{\eta_0} d\eta \, e^{ik\mu(\eta-\eta_0)} \frac{d}{d\eta}\left[\Psi e^{-\tau(\eta)}\right]$$

$\tau(0)$ very large

Just "redefinition" of the monopole which cannot be detected no angular dependence

$$\text{"="} \int_0^{\eta_0} d\eta \, e^{ik\mu(\eta-\eta_0)} \frac{d}{d\eta}\left[\Psi e^{-\tau(\eta)}\right]$$

for our purposes

Since

$$-ik \int_0^{\eta_0} d\eta \, \mu \, \Psi e^{ik\mu(\eta-\eta_0)} e^{-\tau(\eta)} \Rightarrow \int_0^{\eta_0} d\eta \, e^{ik\mu(\eta-\eta_0)} \frac{d}{d\eta}\left[\Psi e^{-\tau(\eta)}\right]$$

This is basically a substitution

$$-ik\mu \longrightarrow \frac{d}{d\eta} \qquad \text{i.e.} \qquad \mu \rightarrow \frac{1}{ik}\frac{d}{d\eta}$$

Note: derivative does NOT act on $e^{ik\mu(\eta-\eta_0)}$

So we can take

$$\Theta_\ell(k,\eta_0) = (-1)^\ell \int_0^{\eta_0} d\eta \, \tilde{S}(k,\eta) e^{-\tau(\eta)} j_\ell[k(\eta-\eta_0)]$$

$(-1)^\ell j_\ell(x) = j_\ell(-x)$

$$= \int_0^{\eta} d\eta \, \tilde{S}(k,\eta) e^{-\tau(\eta)} j_\ell[k(\eta_0-\eta)]$$

and write it as

$$\Theta_\ell(k,\eta_0) = \int_0^{\eta_0} d\eta \, S(k,\eta) j_\ell[k(\eta_0-\eta)]$$

With a redefined source function

$$S(k,\eta) = \tilde{S}(k,\eta,\mu)\, e^{-\tau(\eta)}\Big|_{\mu \to \frac{1}{ik}\frac{d}{d\eta}\,,\ \mu^2 \to -\frac{1}{k^2}\frac{d^2}{d\eta^2}}$$

and therefore

$$S(k,\eta) = e^{-\tau(\eta)}\left[-\dot{\Phi} - \dot{\tau}\left(\Theta_0 + \frac{1}{4}\Pi^{*}\right)\right] \quad \Big\}\ \text{no } \mu \text{ dependence}$$

$$+\frac{d}{d\eta}\left[e^{-\tau(\eta)}\left(\Psi - \frac{i v_b \dot{\tau}}{k}\right)\right] \quad \Big\}\ \propto \mu$$

$$-\frac{3}{4k^2}\frac{d^2}{d\eta^2}\left[e^{-\tau}\dot{\tau}\Pi\right]^{*} \quad \Big\}\ \propto \mu^2$$

* $\tilde{S}$ contains $-\dot{\tau} \times -\frac{1}{2}P_2(\mu)\Pi = \frac{\dot{\tau}}{2}P_2(\mu)\Pi = \frac{\dot{\tau}}{2}\left(\frac{3\mu^2-1}{2}\right)\Pi =$

$= \frac{3}{4}\mu^2\dot{\tau}\Pi - \frac{1}{4}\dot{\tau}\Pi = \dot{\tau}\Pi\left(\frac{3}{4}\mu^2 - \frac{1}{4}\right) \Longrightarrow -\frac{3}{4k^2}\frac{d^2}{d\eta^2}\dot{\tau}\Pi - \frac{\dot{\tau}\Pi}{4}$

Overall source function

$$\boxed{S(k,\eta) = e^{-\tau}\left[-\dot{\Phi} - \dot{\tau}\left(\Theta_0 + \frac{\Pi}{4}\right)\right] + \frac{d}{d\eta}\left[e^{-\tau}\left(\Psi - \frac{i v_b \dot{\tau}}{k}\right)\right] - \frac{3}{4k^2}\frac{d^2}{d\eta^2}\left[e^{-\tau}\dot{\tau}\Pi\right]}$$

Define visibility function

$g(\eta) \equiv -\dot{\tau}e^{-\tau}$

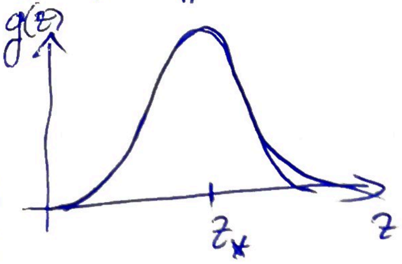
Properties: $\int_0^{\eta_0} d\eta\, g(\eta) = -\int_0^{\eta_0} d\eta \left(\frac{d\tau}{d\eta}\right) e^{-\tau} = -\tau e^{-\tau}\Big|_0^{\eta_0} + \int_0^{\eta_0} d\eta\, \tau \frac{d}{d\eta}\left(e^{-\tau}\right) =$

$$= \underbrace{\cancel{\tau(\eta_0)e^{-\tau(\eta_0)}}}_{\tau(\eta_0)=0} - \underbrace{\cancel{\tau(0)e^{-\tau(0)}}}_{\tau(0)\to\infty} - \int_0^{\eta_0} d\eta\, \tau e^{-\tau} = e^{-\tau}(1+\tau)\Big|_0^{\eta_0} = e^{-\tau(\eta_0)}\left(1+\tau(\eta_0)\right) - e^{-\tau(0)}\left(1+\tau(0)\right)$$

$$\simeq e^{-0}(1+0) = 1 \times 1 = 1!$$

Since $\int d\eta\, g(\eta)=1$, $g(\eta)\equiv -\dot{\tau}e^{-\tau}$ is like a probability density: probability photon last-scattered at η!

$g(\eta)$ suppressed at $\xrightarrow{\eta\to 0}$ because $\tau(\eta)\xrightarrow[\eta\to 0]{}\infty$ $e^{-\tau(\eta)}\to 0$!!)



$\eta \gg \eta_*$ ~~because $e^{\tau(\eta)}$ ... $\tau(\eta)\to 0$~~

because $\tau(\eta)\to 0$ $\left[\tau \sim \int_{\eta}^{\eta_0} d\eta\, a n_e \sigma_T\right]$

Visibility function peaked at $z\sim z_*$ with longer tail at low redshift (It's not symmetrical around z_*)

let's express $S(k,\eta)$ in terms of ~~...~~ $g(\eta)$, also dropping Π

$$S(k,\eta)=e^{-\tau}\left[-\dot{\Phi}-\dot{\tau}\left(\Theta_0+\cancel{\frac{\Pi}{4}}\right)\right]+\frac{d}{d\eta}\left[e^{-\tau}\left(\Psi-\frac{i v_b\dot{\tau}}{k}\right)\right]-\frac{3}{4k^2}\frac{d^2}{d\eta^2}\left[e^{-\tau}\cancel{\dot{\tau}\Pi}\right]$$

$$\simeq -e^{-\tau}\dot{\Phi}\underbrace{-\dot{\tau}e^{-\tau}}_{g(\eta)}\Theta_0\underbrace{-\dot{\tau}e^{-\tau}}_{g(\eta)}\Psi+e^{-\tau}\dot{\Psi}+\frac{d}{d\eta}\left[\underbrace{-\dot{\tau}e^{-\tau}}_{g(\eta)}\frac{i v_b}{k}\right]=$$

$$=g(\eta)\left[\Theta_0(k,\eta)+\Psi(k,\eta)\right]+\cancel{\frac{d}{d\eta}g(\eta)}\frac{d}{d\eta}\left[\frac{i v_b(k,\eta)g(\eta)}{k}\right]+e^{-\tau}\left[\dot{\Psi}(k,\eta)-\dot{\Phi}(k,\eta)\right]$$

Recall $\Theta_\ell \Leftrightarrow S(k,\eta)$ relation.

$$\Theta_\ell(k,\eta_0)=\int_0^{\eta_0} d\eta\, S(k,\eta)\, j_\ell\left[k(\eta_0-\eta)\right]=$$

by parts since $g(\eta_0)\to 0$, $g(0)\to 0$

$$=\int_0^{\eta_0} d\eta\, g(\eta)\left[\Theta_0(k,\eta)+\Psi(k,\eta)\right] j_\ell\left[k(\eta_0-\eta)\right]^{(1)}-\int_0^{\eta_0} d\eta\, g(\eta)\frac{i v_b(k,\eta)}{k}\frac{d}{d\eta} j_\ell\left[k(\eta_0-\eta)\right]^{(2)}$$

$$+\int_0^{\eta_0} d\eta\, e^{-\tau}\left[\dot{\Psi}(k,\eta)-\dot{\Phi}(k,\eta)\right]^{(3)} j_\ell\left[k(\eta_0-\eta)\right]$$

3 terms: (1), (2), (3). Study each one

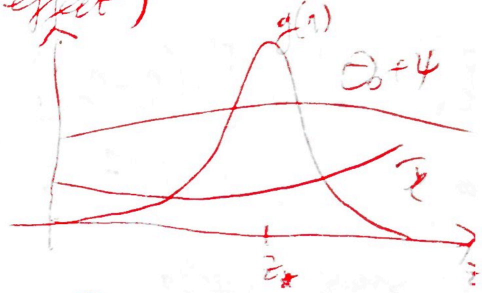
• ③ $$\int_0^{\eta_0} d\eta\, e^{-\tau}\left[\dot{\Psi}(k,\eta)-\dot{\Phi}(k,\eta)\right] j_\ell\left[k(\eta_0-\eta)\right]$$

$e^{-\tau}$ non-negligible only when $\tau \lesssim 1$, well after recombination, when matter dominates and Ψ, Φ constant (both on large & small scales) so $\dot{\Psi}, \dot{\Phi} \sim 0$

③ negligible to zeroth order (but gives ~10% corrections through early & and late integrated Sachs-Wolfe effect)

• ~~② $\int d\eta\, g(\eta)$~~

at a certain ℓ, k $\rightarrow$



• ① $$\int_0^{\eta_0} d\eta\, g(\eta)\left[\Theta_0(k,\eta)+\Psi(k,\eta)\right] j_\ell\left[k(\eta_0-\eta)\right]$$

$g(\eta)$ sharply peaked around η_*

$$\simeq \left[\Theta_0(k,\eta_*)+\Psi(k,\eta_*)\right] j_\ell\left[k(\eta_0-\eta_*)\right] \underbrace{\boxed{\int_0^{\eta_0} d\eta\, g(\eta)}}_{=1} = \left[\Theta_0(k,\eta_*)+\Psi(k,\eta_*)\right] j_\ell\left[k(\eta_0-\eta_*)\right]$$

• ② Similar considerations

$$-\int_0^{\eta_0} d\eta\, g(\eta)\, \frac{i v_b}{k}(k,\eta)\, \frac{d}{d\eta} j_\ell\left[k(\eta_0-\eta)\right] \simeq -\frac{i v_b}{k}(k,\eta_*)\, \frac{d}{d\eta} j_\ell\left[k(\eta_0-\eta_*)\right] \underbrace{\int d\eta\, g(\eta)}_{} =$$

$$= -\frac{i v_b}{k}(k,\eta_*)\, \frac{d}{d\eta} j_\ell\left[k(\eta_0-\eta_*)\right]\Big|_{\eta_*} \underset{\uparrow}{\simeq} 3\Theta_1(k,\eta_*)\left(j_{\ell-1}\left[k(\eta_0-\eta_*)\right] - \frac{(\ell+1)\, j_\ell\left[k(\eta_0-\eta_*)\right]}{k(\eta_0-\eta_*)}\right)$$

$v_b \simeq -3\Theta_1$, $\frac{dj_\ell}{dx} = j_{\ell-1} - \frac{\ell+1}{x} j_\ell$, $\frac{d}{d[k(\eta_0-\eta)]} = -\frac{1}{k}\frac{d}{d\eta}$

In reality also need to account for damping:

$$e^{-\frac{k^2}{k_D(\eta_*)^2}} \longrightarrow \int d\eta\, g(\eta)\, e^{-\frac{k^2}{k_D(\eta)^2}}$$

In summary

$$\Theta_\ell(k,\eta_0) \simeq \left[\Theta_0(k,\eta_*) + \Psi(k,\eta_*)\right] j_\ell\left[k(\eta_0-\eta_*)\right]$$

$$+ 3\Theta_1(k,\eta_*)\left(j_{\ell-1}\left[k(\eta_0-\eta_*)\right] - \frac{(\ell+1)\, j_\ell\left[k(\eta_0-\eta_*)\right]}{k(\eta_0-\eta_*)}\right)$$

$$+ \int_0^{\eta_0} d\eta\, e^{-\tau}\left[\dot{\Psi}(k,\eta) - \dot{\Phi}(k,\eta)\right] j_\ell\left[k(\eta_0-\eta)\right]$$

So to compute Θ_ℓ today we need to know Θ_0, Θ_1, Ψ at recombination

$j_\ell\left[k(\eta_0-\eta_*)\right]$ determines how much anisotropy on an angular scale ℓ^{-1} is contributed by a plane wave with wavenumber k

On small scales

$$\lim_{\ell\to\infty} j_\ell(x) = \frac{1}{\ell}\left(\frac{x}{\ell}\right)^{\ell-\frac{1}{2}} \ll 1 \text{ for } \ell > x \quad \text{i.e. } \ell > k(\eta_0-\eta_*) \approx k\eta_0$$

so $\Theta_\ell(k,\eta_0) \approx 0$ for $\ell > k\eta_0$

small angular scales see little anisotropy from large-wavelength perturbation

~~or equivalently~~ and conversely angular scales $\gtrsim \frac{1}{k\eta_0}$ get little contribution from such a perturbation

So perturbation with wavenumber k mostly contributes to anisotropies on angular scales $\ell \sim k\eta_0 \quad \to k \sim \frac{\ell}{\eta_0}$

Now we need to connect the abstract $\Theta_\ell(k,\eta_0)$ to observed C_ℓ

How do we characterize the temperature field today?

$$T(\vec{x},\hat{p},\eta) = T(\eta)\left[1 + \Theta(\vec{x},\hat{p},\eta)\right]$$

T defined at every point in space and time, but we can only measure it here ($\bar{x}_0$) and now (η_0) : $T(\bar{x}_0, \hat{p}, \eta_0)$

Anisotropies encoded in direction of incoming photons $\hat{p}$

CMB maps labelled not as $\hat{p}_x, \hat{p}_y, \hat{p}_z$, but in polar coordinates (θ, φ) such that $\hat{p}_z = \cos\theta$, $\hat{p}_x = \sin\theta\cos\varphi$, $\hat{p}_y = \sin\theta\sin\varphi$

Let's stick with $\hat{p}$, expand Θ in spherical harmonics (since observed on the sky)

$$\Theta(\bar{x}, \hat{p}, \eta) = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} a_{lm}(\bar{x}, \eta) Y_{lm}(\hat{p})$$

$$l, m \xleftrightarrow{\text{CONJUGATE TO}} \hat{p}$$

$$\bar{k} \longleftrightarrow \bar{x}$$

Y_{lm} : generalize Fourier transform on sphere, complete set of eigenfunctions replacing $e^{i\bar{k}\cdot\bar{x}}$ for flat space

So all information contained in full $T(\hat{p})$ is also contained in a_{lm}

Given a certain angular resolution σ_θ this corresponds to $l_{max} \sim \frac{\pi}{\sigma_\theta}$ given by total number of recoverable a_{lm}

$$\sum_{l=0}^{l_{max}} (2l+1) = (l_{max}+1)^2 = \frac{4\pi \text{ rad}^2}{\sigma_\theta [\text{rad}^2]}$$ (e.g. COBE $\sigma_\theta \sim 7°$, $l_{max} \simeq 3$

So basically the observables are the a_{lm}s : how can they be related to $\Theta_l(k, \eta_0)$? Use orthogonality property of spherical harmonics:

$$\int d\Omega \, Y_{lm}(\hat{p}) Y^*_{l'm'}(\hat{p}) = \delta_{ll'}\delta_{mm'}$$

$$\Theta(\bar{x},\hat{p},\eta) = \sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell} a_{\ell m}(\bar{x},\eta)\, Y_{\ell m}(\hat{p})$$

$$\int d\Omega\, Y_{\ell m}(\hat{p})\, Y^{*}_{\ell' m'}(\hat{p}) = \delta_{\ell\ell'}\delta_{mm'}$$

$$\Downarrow$$

$$a_{\ell m}(\bar{k},\eta) \stackrel{?}{=} \int d\Omega\, Y^{*}_{\ell m}(\hat{p})\,\Theta(\bar{k},\hat{p},\eta) = \int d\Omega \sum_{\ell'}\sum_{m'} a_{\ell m} Y_{\ell' m'} Y^{*}_{\ell m} =$$

$$= \cancel{\int d\Omega} \sum_{\ell'}\sum_{m'} \delta_{\ell\ell'}\delta_{mm'}\, a_{\ell m} = a_{\ell m} \qquad \text{QED}$$

Ultimately we want $a_{\ell m}(\bar{x},\eta)$ so it's really

$$a_{\ell m}(\bar{x},\eta) = \int \frac{d^3k}{(2\pi)^3}\, e^{i\bar{k}\cdot\bar{x}} \int d\Omega\, Y^{*}_{\ell m}(\hat{p})\,\Theta(\bar{k},\hat{p},\eta)$$

Same as density perturbations, we cannot make prediction for $a_{\ell m}$, just about their distribution (related to inflationary fluctuations)

$$\langle a_{\ell m}\rangle = 0 \qquad \langle a_{\ell m} a^{*}_{\ell' m'}\rangle = \delta_{\ell\ell'}\delta_{mm'} \boxed{C_\ell} \rightarrow \text{what we measure}$$

~~For each l,~~ For a given ℓ, each $a_{\ell m}$ has the same variance ($-\ell \le m \le \ell$)
So measuring the $(2\ell+1)$ $a_{\ell m}$'s for each ℓ means sampling the distribution
There is a fundamental limitation to how well we can measure the C_ℓ's

$\Downarrow$

COSMIC VARIANCE (most pronounced at low ℓ)

$$\left(\frac{\Delta C_\ell}{C_\ell}\right)\Bigg|_{CV} \sim \text{inverse of square root of number of samples} = \sqrt{\frac{2}{2\ell+1}}$$

But now we want to relate C_ℓ ~~to~~ $\Theta_\ell(k,\eta_0)$

Implicitly everything depends on η from now on

$\cancel{a_{lm}(\vec{x},\eta) = \int d^3}$

$$a_{lm} = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \int d\Omega\, Y^*_{lm}(\hat{p})\,\Theta(\vec{k},\hat{p},\eta)$$

$\langle a_{lm} a^*_{l'm'}\rangle \sim \langle \Theta(\vec{k},\hat{p})\,\Theta^*(\vec{k}',\hat{p}')\rangle$

$$\langle a_{lm} a^*_{l'm'}\rangle = \delta_{ll'}\delta_{mm'} C_l$$

Separate these phenomena: Inflation sets initial amplitude and phase (1); evolution turns perturbations into anisotropies (2)

$$\Theta = \underbrace{\delta}_{\text{no } \hat{p} \text{ dependence}} \underbrace{\frac{\Theta}{\delta}}_{\text{related to transfer function}}$$

no dependence on initial amplitude → removed from the averaging of the distribution

$$\Rightarrow \langle \Theta(\vec{k},\hat{p})\,\Theta(\vec{k}',\hat{p}')\rangle = \langle \delta(\vec{k})\,\delta^*(\vec{k}')\rangle \frac{\Theta(\vec{k},\hat{p})}{\delta(\vec{k})} \frac{\Theta^*(\vec{k}',\hat{p}')}{\delta(\vec{k}')} =$$

$$= (2\pi)^3 \delta^3(\vec{k}-\vec{k}')\,P(k) \boxed{\frac{\Theta(k,\hat{k}\cdot\hat{p})}{\delta(k)} \frac{\Theta^*(k,\hat{k}\cdot\hat{p}')}{\delta^*(k)}}$$

→ only depends on k, $\hat{k}\cdot\hat{p}$; two modes with these quantities identical will evolve identically

Squaring a_{lm} we see

$$C_l = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \int d\Omega\, \cancel{(2\pi)^3}\, \delta^3(\vec{k}-\vec{k}')\,P(k)\,Y^*_{lm}(\hat{p}) \frac{\Theta(k,\hat{k}\cdot\hat{p})}{\delta(k)} \int \frac{d^3k'}{\cancel{(2\pi)^3}} e^{i\vec{k}'\cdot\vec{x}} \int d\Omega'\, Y_{lm}(\hat{p}') \frac{\Theta^*(k,\hat{k}'\cdot\hat{p}')}{\delta^*(k)}$$

use $\delta^3(\vec{k}-\vec{k}')$

$$= \int \frac{d^3k}{(2\pi)^3} P(k) \int d\Omega\, Y^*_{lm}(\hat{p}) \frac{\Theta(k,\hat{k}\cdot\hat{p})}{\delta(k)} \int d\Omega'\, Y_{lm}(\hat{p}') \frac{\Theta^*(k,\hat{k}\cdot\hat{p}')}{\delta^*(k)}$$

Now expand $\Theta(k,\hat{k}\cdot\hat{p})$ and $\Theta(k,\hat{k}\cdot\hat{p}')$ using

$$\Theta(k,\hat{k}\cdot\hat{p}) = \sum_l (-i)^l (2l+1)\, P_l(\hat{k}\cdot\hat{p})\,\Theta_l(k)$$

$$\Rightarrow C_\ell = \int \frac{d^3k}{(2\pi)^3} P(k) \sum_{\ell' \ell''} (-i)^{\ell'} (-i)^{\ell''} (2\ell'+1)(2\ell''+1) \frac{\Theta_{\ell'}(k)\,\Theta^*_{\ell''}(k)}{|\delta(k)|^2}$$

$$\underbrace{\int d\Omega\, P_{\ell'}(\hat{k}\cdot\hat{p})\, Y^*_{\ell m}(\hat{p})}_{\delta_{\ell\ell'} \frac{4\pi}{2\ell+1} Y^*_{\ell m}} \underbrace{\int d\Omega'\, P_{\ell''}(\hat{k}\cdot\hat{p}')\, Y_{\ell m}(\hat{p}')}_{\delta_{\ell''\ell} \frac{4\pi}{2\ell+1} Y_{\ell m}} =$$

$$= \frac{\cancel{16\pi^2}^{\,2}}{\cancel{8\pi^3}} \int dk\, k^2 P(k) \left| \frac{\Theta_\ell(k)}{\delta(k)} \right|^2 \frac{\cancel{(2\ell+1)^2}}{\cancel{(2\ell+1)^2}} \underbrace{\int d\Omega_k\, |Y_{\ell m}|^2}_{=1}$$

$$\Rightarrow C_\ell = \frac{2}{\pi} \int_0^\infty dk\, k^2 P(k) \left| \frac{\Theta_\ell(k)}{\delta(k)} \right|^2$$

Knowing $\Theta_\ell(k, \eta_0)\left[\Theta_0(k,\eta_*), \Theta_1(k,\eta_*), \Phi(k,\eta_*)\right]$ we can plot C_ℓ

Variance of $a_{\ell m}$ ~ integral over Fourier modes of variance of $\Theta_\ell(k)$

<u>Anisotropy spectrum today</u>

Start from

$$\Theta_\ell(k,\eta_0) \simeq \left[\Theta_0(k,\eta_*) + \Psi(k,\eta_*)\right] j_\ell[k(\eta_0-\eta_*)]$$
$$+ 3\Theta_1(k,\eta_*)\left(j_{\ell-1}[k(\eta_0-\eta_*)] - \frac{(\ell+1)\, j_\ell[k(\eta_0-\eta_*)]}{k(\eta_0-\eta_*)} \right)$$
$$+ \int_0^{\eta_0} d\eta\, e^{-\tau}\left[\dot{\Psi}(k,\eta) - \dot{\Phi}(k,\eta)\right] j_\ell[k(\eta_0-\eta)]$$

Large-angle anisotropies: not affected by causal microphysics, at recombination only monopole contributes

Large-angle anisotropy:

recombination far enough after matter-radiation equality that we can use $D(a)$

$$\Theta_0(\eta_*)+\Psi(\eta_*) \simeq \frac{\Psi(\eta_*)}{3} \simeq$$

$$\simeq \frac{1}{3D_1(a=1)}\Psi(\eta_0) = -\frac{1}{3D_1(a=1)}\Phi(\eta_0) \qquad [\Phi \simeq -\Psi \text{ at late times}]$$

Recall from Poisson equation (large-k, no radiation)

$$\delta(k,a) \simeq \frac{k^2\Phi a}{\frac{3}{2}\Omega_m H_0^2} \Longrightarrow \Theta_0(\eta_*)+\Psi(\eta_*) \simeq -\frac{\Phi(\eta_0)}{3D_1(a=1)} \simeq -\frac{\Omega_m H_0^2\,\delta(\eta_0)}{2k^2 D_1(a=1)}$$

Then we can get the low-ell C_ℓ

$$C_\ell = \frac{2}{\pi}\int_0^\infty dk\, k^2 P(k)\left|\frac{\Theta_\ell(k)}{\delta(k)}\right|^2 \simeq \frac{\Omega_m^2 H_0^4}{2\pi D_1^2(a=1)}\int_0^\infty \frac{dk}{k^2} J_\ell^2[k(\eta_0-\eta_*)]\,P(k)$$

$\hookrightarrow C_\ell^{SW}$: Sachs-Wolfe effect

$$P(k,a) \underset{\text{large scales}}{\simeq} 2\pi^2\delta_H^2 \frac{k^n}{H_0^{n+3}}\underbrace{T^2(k)}_{\simeq 1}\left(\frac{D_1(a)}{D_1(a=1)}\right)^2$$

$$\Rightarrow C_\ell^{SW} \simeq \pi H_0^{1-n}\left(\frac{\Omega_m}{D_1(a=1)}\right)^2\delta_H^2\int_0^\infty \frac{dk}{k^{2-n}} J_\ell^2[k(\eta_0-\eta_*)]$$

[illegible] $\ll \eta_0$ $\quad dk=\frac{1}{\eta_0}dx$

$x \equiv k\eta_0$

$$\simeq 2^{n-4}\pi^2(\eta_0 H_0)^{1-n}\left(\frac{\Omega_m}{D_1(a=1)}\right)^2\delta_H^2\underbrace{\int_0^\infty \frac{dx}{x^{2-n}}J_\ell^2(x)}$$

$$= \frac{\Gamma\left(\ell+\frac{n}{2}-\frac{1}{2}\right)}{\Gamma\left(\ell+\frac{5}{2}-\frac{n}{2}\right)}\frac{\Gamma(3-n)}{\Gamma^2\left(2-\frac{n}{2}\right)} \xrightarrow{n=1} \frac{1}{\ell(\ell+1)}\frac{4}{\pi}$$

$$\Rightarrow C_\ell^{SW} \simeq \frac{\pi}{2\ell(\ell+1)}\left(\frac{\Omega_m}{D_1(a=1)}\right)^2\delta_H^2$$

$\Longrightarrow \boxed{\ell(\ell+1)C_\ell^{SW} = \text{const}}$

→ typically we plot $\ell(\ell+1)$, to see the "Sachs-Wolfe "plateau"

The plateau is actually not completely flat because of:

- Θ_1 contributes
- $\dot{\Phi}, \dot{\Psi}$ non-zero when DE enters, late integrated Sachs-Wolfe effect

If $n \neq 1$, $C_\ell^{SW} \sim \ell^{n-1}$ $\left[\text{as } \int \frac{dx}{x^{2-n}} J_\ell^2(x) \sim \int \frac{d\ell}{\ell^{2-n}}\right]$ (peaks at $x \sim \ell$)

So already rough measurement of Sachs-Wolfe Plateau excludes large deviations from $n_s \sim 1$

<u>Small scales</u>

Depend on monopole, dipole, and ISW effect

- <u>Monopole</u>: gives most of final anisotropy spectrum structure. Note:
  - * monopole "zeros" not zeros but troughs due to the fact that not just one mode contributes to anisotropy at a given angular scale
  - * Peak position slightly wrong compared to $\ell = k\eta_0$, rather slightly smaller ℓ! $J_\ell(x)$ vs ℓ peaks slightly below x ($\sim 0.9x$)
  - * $\ell_p \simeq 0.75\, \pi \frac{\eta_0}{r_s}$

<u>Dipole</u>: smaller than monopole, out of phase

- * raises overall anisotropy, fills in the troughs precisely because it is out of phase (lowers peak-to-trough height ratio)
- * Θ_0 and Θ_1 add incoherently (so dipole contributes less than what one would think)

- ~~Late~~ ISW effect:

  * important if potential changes after recombination
  * residual radiation at recombination since matter-radiation transition is gradual
* assume potential changes at ~~state~~ time η_c, affects all scales $k\eta_c > 1$, Bessel function peaks at $\ell \sim k(\eta_0 - \eta_c)$ so all scales $\ell > \frac{\eta_0 - \eta_c}{\eta_c}$ affected
* Early ISW effect adds coherently with monopole

$$\Theta_\ell^{eISW}(k,\eta_0) = \int_0^{\eta_0} d\eta\, e^{-\tau}\left[\dot{\Psi}(k,\eta) - \dot{\Phi}(k,\eta)\right] J_\ell[k(\eta_0-\eta)]$$

$$\cancel{= \left[\Psi - \Phi\right]\Big|_0^{\eta_0} J_\ell[k(\eta_0-\eta)]\Big|_0^{\eta_0} + \text{subdominant}}$$

~~$\simeq$ dominant contribution from $\eta \sim \eta_*$~~

picks up most important contributions at $\eta \sim \eta_*$

$$\Rightarrow \Theta_\ell^{eISW}(k,\eta_0) \simeq \left[\Psi(k,\eta_0) - \Psi(k,\eta_*) - \Phi(k,\eta_0) + \Phi(k,\eta_*)\right] J_\ell\left[k(\eta_0 - \eta_*)\right]$$

Recall monopole

$$\left[\Theta_0(k,\eta_*) + \Psi(k,\eta_*)\right] J_\ell\left[k(\eta_0 - \eta_*)\right]$$

↳ adds in phase with the monopole, same Bessel function

↳ Boosts first peak

~~Late~~ ISW effect:

* important if potential changes after recombination
* residual radiation at recombination since matter-radiation transition is gradual
* assume potential changes at ~~state~~ time η_c, affects all scales $k\eta_c > 1$, Bessel function peaks at $l \sim k(\eta_0 - \eta_c)$ so all scales $l > \frac{\eta_0 - \eta_c}{\eta_c}$ affected
* Early ISW effect adds coherently with monopole

$$\Theta_l^{eISW}(k,\eta_0) = \int_0^{\eta_0} d\eta\, e^{-\tau} \left[\dot{\Psi}(k,\eta) - \dot{\Phi}(k,\eta) \right] J_l\left[k(\eta_0 - \eta)\right]$$

$$\cancel{= \left[\Psi - \Phi\right]\Big|_0^{\eta_0} J_l\left[k(\eta_0-\eta)\right]\Big|_0^{\eta_0} + \text{subdominant}}$$

$\simeq$ dominant contribution from $\eta \sim \eta_*$

picks up most important contributions at $\eta \sim \eta_*$

$$\Rightarrow \Theta_l^{eISW}(k,\eta_0) \simeq \left[\Psi(k,\eta_0) - \Psi(k,\eta_*) - \Phi(k,\eta_0) + \Phi(k,\eta_*)\right] J_l\left[k(\eta_0 - \eta_*)\right]$$

Recall monopole

$$\left[\Theta_0(k,\eta_*) + \Psi(k,\eta_*)\right] J_l\left[k(\eta_0 - \eta_*)\right]$$

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Cosmological parameters

Possible parameters:

- $\Omega_k = 1 - \Omega_m - \Omega_\Lambda$
- $C_{10} \longrightarrow A_s$
- n_s
- r
- $\tau\,(z_{re})$
- $\Omega_b h^2 \longrightarrow \omega_b$
- $\Omega_m h^2 \longrightarrow \omega_m$
- Ω_Λ

6 ΛCDM parameters:

- ω_b
- ω_c
- $\theta_s \longleftrightarrow H_0$
- A_s
- n_s
- τ

Note: in a flat Universe $\Omega_m + \Omega_\Lambda \simeq 1$ (neglect radiation today!)

$$\Omega_m + \Omega_\Lambda = 1 \Rightarrow \Omega_m h^2 + \Omega_\Lambda = h^2 \Rightarrow \frac{\Omega_m h^2}{h^2} + \Omega_\Lambda = 1$$

Vary $(\Omega_m h^2, \Omega_\Lambda)$ or (h, Ω_Λ)?

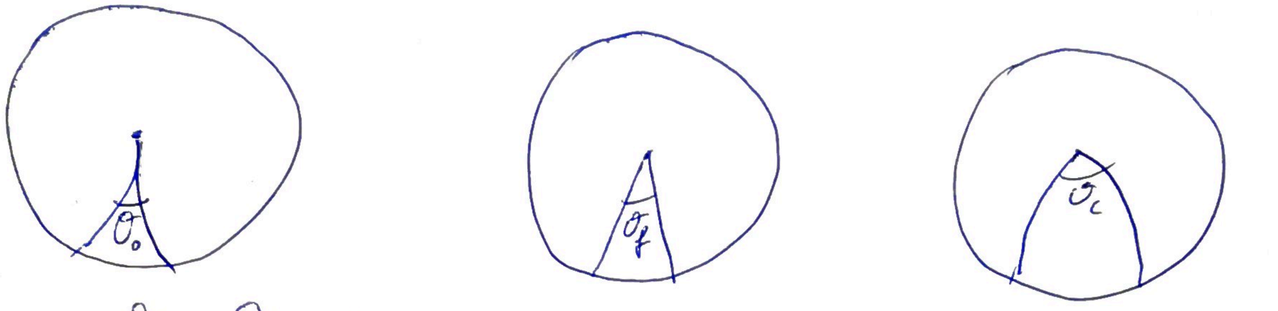
$(\Omega_m h^2, \Omega_\Lambda)$ $\downarrow$ raise Ω_Λ fixing $\Omega_m h^2$, z_{eq} fixed

(h, Ω_Λ) $\downarrow$ raise Ω_Λ but lower Ω_m to keep $\Omega_m h^2 + \Omega_\Lambda = h^2$ while fixing h, so z_{eq} is lower ($\Omega_m h^2$ lower)

- Curvature

Open Universe $\rightarrow$ smaller angles $\longrightarrow$ peaks to ~~smaller~~ larger ℓ

Closed Universe $\rightarrow$ larger angles $\longrightarrow$ peaks to ~~larger~~ smaller ℓ



$\theta_0 < \theta_f < \theta_c$

peak position depends on $\quad l_p \propto \frac{\eta_0}{r_s} \propto \left[\int \frac{dz}{H(z)}\right]$

$\left[\, l = k\eta_0 \rightarrow l_p = k_p \eta_0 = \frac{n\pi\eta_0}{r_s} \,\right]$

$\cancel{H(z) = H_0\sqrt{\Omega_m(1+z)^3 + \Omega_\Lambda + \Omega_k(1+z)^2}}$ ~~Closed~~ $\cancel{\Omega_k < 0 \quad \eta_0}$

Angular diameter distance to last-scattering larger in an open Universe

$D_A \propto (1-\Omega_k)^{-0.45}$

• Degenerate parameters

- C_{10} moves spectrum up and down
- n_s if <1 suppresses small relative to large scales

$$\frac{C_\ell(n_s)}{C_\ell(n_s=1)} \simeq \left(\frac{\ell}{\ell_{pivot}}\right)^{n_s-1} \qquad P(k) \propto k^{n_s}$$

- τ

$$\underbrace{T(1+\Theta)e^{-\tau}}_{\text{escape}} + \underbrace{T(1-e^{-\tau})}_{\text{from reionized region}} = T(1+\Theta e^{-\tau}) \Rightarrow \Theta e^{-\tau}$$

suppresses all scales within horizon at reionization

- r Enhances anisotropies for $\ell < 100$

• Distinct Imprints

• $\Omega_b h^2$: shifts peaks due to shift in r_s, affects odd-to-even peak heig[hts]
changes diffusion scale

• Ω_Λ: late ISW effect. enhances ~~small~~ large-scale anisotropies

• $\Omega_m h^2$: if lowered, enhances early ISW effect